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The unitary representations of the general covariant group algebra

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Abstract. A new series of unitary representations (UR) for the Lie algebra of the general covariant group in N-dimensional real vector space (group Diff R^N) is constructed. The matrix representations for the generators of the algebra and the relationship between the UR for the group Diff R^N algebra and the non-linear realisation of the group Diff R^N algebra are found.

1. Introduction

The infinite-parameter Lie groups represent the basis of the numerous dynamical symmetries which are useful in particle physics. The general covariance in the theory of gravity, the isotopic transformations in the Yang–Mills theory, and the local gauge transformations in electrodynamics are all examples of such dynamical symmetries.

The purpose of this paper is to examine the unitary representations (UR) of general covariant group algebra in N-dimensional real vector space (algebra of Diff R^N group). The transformations of the Diff R^N group have the form:

$$x'_\mu = x'_\mu(x_1, x_2, \ldots, x_N) \tag{1}$$

where $x'_\mu(x)$ are the arbitrary differentiable functions of coordinates in the space R^N; moreover the transformation (1) must be invertible. The UR of the general covariant group in four-dimensional space–time are useful in finding the Diff R^N group algebra at high energies, and also in constructing the renormalisable theory of gravity, etc.

The finite-dimensional non-unitary representations of the algebra of the Diff R^N group are well known. These representations are realised by matrices which depend on the coordinates x_μ, and on the tensors which are the non-unitary representations of the linear group GL(N, R). The law, by the infinitesimal transformation of coordinates of the tensor components $\Phi_\alpha(x)$, is

$$\Phi'_\alpha(x') = \Phi_\alpha(x) + \mathrm{i}\epsilon\frac{\partial f_\mu}{\partial x_\rho}(T_{\mu\rho})_{\alpha\beta}\Phi_\beta(x); \qquad x'_\mu = x_\mu + \epsilon f_\mu(x) \quad |\epsilon| \ll 1. \tag{2}$$

The matrices $T_{\mu\rho}$ realise the finite-dimensional representations of the group GL(N, R). For example, the covariant vector has components $\Phi_\alpha(x)$ ($\alpha = 1, 2, \ldots, N$) and $(T_{\mu\rho})_{\alpha\beta} = -\mathrm{i}\delta_{\rho\beta}\delta_{\mu\alpha}$. The representation (2) is non-unitary, because the invariant scalar product does not exist. The importance of the representation (2) in physics and geometry is well known.

A number of papers (see the review by Vershik *et al* 1975) have investigated some UR of the Diff R^N group. It is shown in the present paper that new series of the algebra of the group exist which are of interest for physical applications. They are defined in the infinite-dimensional space of the UR of the group GL(N, R).

2. The unitary representations of Lie algebra of Diff R^N group

Let U be the set of functions on R^N such that the transformation

$$x'_\mu = x_\mu + \epsilon f_\mu(x) \qquad |\epsilon| \ll 1 \tag{3}$$

is the infinitesimal transformation of the Diff R^N group for any $f_\mu(x) \in U$. Let Diff R^N denote the Lie algebra of the Diff R^N group. Let T_f and $T_h(f, h \in U)$ be the elements of diff R^N whose differential operator representations are given by

$$T_f = -f_\mu \frac{\partial}{\partial x_\mu}, \qquad T_h = -h_\mu \frac{\partial}{\partial x_\mu}. \tag{4}$$

The T_f, T_h have the following commutation relations:

$$[T_f, T_h] = T_\gamma \tag{5}$$

where the element T_γ behaves like the differential operator

$$T_\gamma = \left(f_\nu \frac{\partial}{\partial x_\nu} h_\mu - h_\nu \frac{\partial}{\partial x_\nu} f_\mu \right) \frac{\partial}{\partial x_\mu}. \tag{6}$$

Let An Diff R^N be the group of analytic diffeomorphisms of R^N. First we shall find the UR of An Diff R^N. Let us expand arbitrary analytic functions f_μ^{an} of the infinitesimal transformation of An Diff R^N:

$$x'_\mu = x_\mu + \epsilon f_\mu^{\text{an}}(x), \qquad |\epsilon| \ll 1$$

as an infinite series in powers of the coordinates

$$f_\mu^{\text{an}} = c_\mu + \sum_{n=1}^{\infty} c_{\mu\nu_1\nu_2\ldots\nu_n} x_{\nu_1} x_{\nu_2} \ldots x_{\nu_n}. \tag{7}$$

The coefficients of the series are the parameters of An Diff R^N. Let T_f^{an} be the element of An diff R^N where the differential representation is given by

$$T_f^{\text{an}} = -f_\mu^{\text{an}} \frac{\partial}{\partial x_\mu}. \tag{8a}$$

Then we have

$$T_f^{\text{an}} = \mathrm{i} c_\mu P_\mu + \mathrm{i} \sum_{n=1}^{\infty} c_{\mu\nu_1\nu_2\ldots\nu_n} F_{\mu\nu_1\nu_2\ldots\nu_n} \tag{8b}$$

where the generators P_μ, $F_{\mu\nu_1\nu_2\ldots\nu_n}$ behave like the operators

$$P_\mu = \mathrm{i}\frac{\partial}{\partial x_\mu}, \qquad F_{\mu\nu_1\nu_2\ldots\nu_n} = \mathrm{i} x_{\nu_1} x_{\nu_2} \ldots x_{\nu_n} \frac{\partial}{\partial x_\mu}. \tag{9}$$

We note two important subalgebras of An diff R^N, namely, the algebra sL(N, R) of the special linear group SL(N, R), and the algebra c(N) of the conformal group C(N).

sL(N, R) has the basis consisting of the generators $M_{\mu\nu}=F_{\mu\nu}-F_{\nu\mu}$ of the rotation group and the generators $R_{\mu\nu}=F_{\mu\nu}+F_{\nu\mu}-(2/N)\delta_{\mu\nu}F_{\gamma\gamma}$ of the special linear transformations. c(N) includes the translation generators P_μ, the generators $M_{\mu\nu}$ and the generators of the scale and special conformal transformations $D=F_{\gamma\gamma}$ and $K_\mu=F_{\mu\gamma\gamma}-2F_{\gamma\gamma\mu}$. The generators of c($N$) and sL($N$, R) do not generate a finite algebra. They generate (Ogievetsky 1973) the algebra An diff R^N.

Let us describe more fully the unitary single valued representations of SL(N, R) which will be used below. We shall consider SL(N, R) as a group of $N\times N$ matrices. The UR of SL(N, R) may be found by the method of induced representation (Mackey 1963). Accordingly the Iwasawa decomposition (Helgason 1962) $G=\mathrm{SL}(N, R)$ can be written as a product

$$G=K\,.\,A\,.\,T \tag{10}$$

where T, A and K are subgroups of G. T is a nilpotent subgroup of SL(N, R) and its elements are upper triangular matrices; $K=\mathrm{SO}(N)$ is the maximal compact subgroup of SL(N, R); and A is the group of diagonal matrices (with positive elements). Let M be the centraliser of A in K. M is the group of all diagonal matrices with entries ± 1 on the diagonal. The representations of the principal series are obtained as the induced representations on the homogeneous space G/G', where $G'=M\,.\,A\,.\,T$ (Warner 1972). There is an equivalent way of defining the representations of the principal series. Namely, let σ be an irreducible UR of M and H^σ be the set of all square-integrable functions on K with respect to the invariant measure dk on K such that, for each $m\in M$, $f(km)=\sigma(m)f(k)$ for every $k\in K$. For arbitrary $g\in\mathrm{SL}(N, R)$ and $k\in K$ there exists a unique decomposition of the element gk

$$gk=k_g\,\mathrm{e}^{h(g,k)}t; \qquad k_g\in K,\ \mathrm{e}^{h(g,k)}\in A,\ t\in T \tag{11}$$

where $h(g, k)$ denotes an element of the Lie algebra corresponding to A. The group A has the generators $A_1, A_2, \ldots, A_{N-1}$ and if $t_1, t_2, \ldots, t_{N-1}$ are the corresponding group parameters, one has $h(g, k)=\Sigma_{i=1}^{N-1} t_i(g, k)A_i$. Let α be a linear function such that $\alpha(\Sigma_{i=1}^{N-1} t_iA_i)=\Sigma_{i=1}^{N-1} t_i\alpha(A_i)$. It is possible then to define a representation $T(g)$ ($g\in\mathrm{SL}(N, R)$) on H^σ in the following way (Harish-Chandra 1953):

$$T(g)f(k)=\{\exp[\alpha(h(k, g^{-1}))]\}f(k_{g^{-1}})(\mathrm{d}k_{g^{-1}}/\mathrm{d}k)^{1/2}. \tag{12}$$

In Dirac bra–ket notation $f(k)=\langle k|f\rangle$ and the representation defined above reads

$$\langle k|T(g)|f\rangle=\langle k|f'\rangle=(\mathrm{d}k_{g^{-1}}/\mathrm{d}k)^{1/2}\{\exp[\alpha(h(k, g^{-1}))]\}\langle k_{g^{-1}}|f\rangle.$$

If the linear function α takes purely imaginary values on the Lie algebra corresponding to A, the representation (12) is unitary to the scalar product

$$\langle f|f\rangle=\int f^*(k)f(k)\,\mathrm{d}k. \tag{13}$$

Let Ω be the set of all eigenvalence classes of unitary irreducible representations (UIR) of SO(N). We denote by V_ω the space of UIR of SO(N) for any $\omega\in\Omega$. Then an arbitrary function $f(k)(k\in\mathrm{SO}(N))$ can be written as

$$f(k)=\sum_{\omega\in\Omega}\sum_{i,j}^{d_\omega} f_{ij}^\omega t_{ij}^\omega(k)d_\omega^{1/2} \tag{14}$$

where $f_{ij}^\omega\in C$, $d_\omega=\dim V_\omega$ and $t_{ij}^\omega(k)$ are the matrix elements of the UIR of SO(N) on V_ω. Let $\rho_i(i=1, 2, \ldots, d_\omega)$ be the basis of V_ω such that $t_{ij}^\omega(m)=t_j^\omega(m)\delta_{ij}$. Let

$S_1, S_2, \ldots, S_{p_\omega}$ be the indices for which $\sigma(m)=t_j^\omega(m)$ $(j=S_1,\ldots,S_{p_\omega})$.Then using the equation: $f(km)=\sigma(m)f(k)(f(k)\in H^\sigma)$ we obtain the decomposition $f(k)\in H^\sigma$ in the following form:

$$f(k)=\sum_{\omega\in\Omega}\sum_{j=S_1}^{S_{p_\omega}}\sum_{i=1}^{d_\omega} f_{ij}^\omega t_{ij}^\omega(k)d_\omega^{1/2}. \tag{15}$$

The representation of the principal series $T(g)$ (12) can now be conveniently given as a set of matrix elements in the discrete basis $|{}^\omega_{ij}\rangle$, where $\langle k|{}^\omega_{ij}\rangle=d_\omega^{1/2}t_{ij}^\omega(k)$. We obtain

$$T(g)|{}^\omega_{ij}\rangle=\sum_{\omega'i'j'}\int \mathrm{d}k t_{i'j'}^{*\omega'}(k)t_{ij}^\omega(k_{g^{-1}})\left(\frac{\mathrm{d}k_{g^{-1}}}{\mathrm{d}k}\right)^{1/2}\exp[\alpha(h(k,g^{-1})](d_\omega d_{\omega'})^{1/2}|{}^{\omega'}_{i'j'}\rangle. \tag{16}$$

The principal series of representations do not exhaust all the UR of $\mathrm{SL}(N,R)$. A complete list of the UR of $\mathrm{SL}(N,R)$ is established by Gelfand and Graev (1956). It is convenient however for physical applications to have the UR of $\mathrm{SL}(N,R)$ in the space functions on $\mathrm{SO}(N)$. The realisation of all UR of Gelfand and Graev in the space functions on the maximal compact subgroup is known only for $\mathrm{SL}(2,R)$ (Bargmann 1947, Gelfand *et al* 1966), and $\mathrm{SL}(3,R)$ (Borisov 1974, Sijacki 1975). In the discrete basis $|m\rangle$ (m is the integer number) the UR of $\mathrm{SL}(2,R)$ have the form

$$\begin{aligned}&T(g)|m\rangle=\sum_n\int \mathrm{d}k\, f_n^*(k)(a_{22}(k,g^{-1}))^{s-1}f_m(k_{g^{-1}})|m\rangle\\ &g^{-1}k=k_{g^{-1}}a(k,g^{-1}); \qquad k,k_{g^{-1}}\in\mathrm{SO}(2),\quad g\in\mathrm{SL}(2,R)\end{aligned} \tag{17}$$

where

$$a=\mathrm{diag}(a_{11},a_{22})\in A, \qquad k=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},$$

$$f_n(k)=\exp in\theta, \qquad 0\leqslant\theta\leqslant 2\pi.$$

There are three series of UR of $\mathrm{SL}(2,R)$ according to the classification of Gelfand and Graev (1956): the principal series ($s=i\rho$, $\rho\in R$, $m=0,\pm1,\pm2,\ldots$), the supplementary series ($-1<s<1$, $m=0,\pm1,\pm2,\ldots$), and the discrete series ($s=0,-1,-2,\ldots$; $|m|=-s+1,-s+3,\ldots$). If $f(\theta)=\Sigma_m f_m\,\mathrm{e}^{im\theta}$ the representation (17) is unitary with respect to the scalar product

$$\langle f|f\rangle=\sum_m |f_m|^2N(m,s) \tag{18}$$

where the coefficients $N(m,s)$ were obtained by Bargmann (1947) in the obvious form.

Let us describe three series of the UR of $\mathrm{SL}(3,R)$. Let $D_{nm}^l(k)$ ($k\in\mathrm{SO}(3)$) be the rotation matrix element corresponding to an angular momentum l, $-l\leqslant m,n\leqslant l$. The UIR of $\mathrm{SL}(3,R)$ are defined in the $|lnm\rangle$ ($l=0,1,2,\ldots$) basis as follows (Sijacki 1975)

$$\begin{aligned}T(g)|lnm\rangle=\sum_{l'm'n'}&\int \mathrm{d}k\,\tau_{n'm'}^{*l'}(k)(a_{11}(k,g))^\mu(a_{22}(k,g))^\lambda\\ &\times[(2l+1)(2l'+1)]^{1/2}\tau_{nm}^l(k_g)|l'n'm'\rangle.\end{aligned} \tag{19}$$

Here the decomposition

$$kg=ta(k,g)k_g; \qquad k,k_g\in\mathrm{SO}(3),\quad a(k,g)\in A,\quad t\in T$$

is used and we have the following series.

(i) Principal series:

$$\tau^l_{nm}(k)=\frac{1}{\sqrt{2}}(D^l_{nm}(k)\pm(-1)^{l-n}D^l_{-nm}(k) \quad (n\geqslant 0)$$

$$\mathrm{i}(\mu+2)\in R, \qquad \mathrm{i}(\lambda+1)\in R.$$

(ii) Supplementary series:

(a) $$\tau^l_{nm}(k)=\frac{1}{\sqrt{2}}(D^l_{nm}(k)\pm(-1)^{l-n}D^l_{-nm}(k)) \quad (n\geqslant 0),$$

$$\mu+\lambda=-3+\mathrm{i}\sigma_2, \qquad \mu-\lambda=-1+\delta_1, \quad \sigma_2, \delta_2\in R, |\delta_1|<1;$$

(b) $$\tau^l_{nm}(k)=\frac{1}{\sqrt{2}}(D^l_{nm}(k)\pm(-1)^{l-n}D^l_{-nm}(k)) \quad (n\geqslant n_0),$$

$$\mu+\lambda=-3+\mathrm{i}\sigma_2, \qquad \sigma_2\in R, \mu-\lambda=-n_0=-1,-2,\ldots, n=n_0(\mathrm{mod}\ 2)$$

(iii) Multiplicity free representations:

$$\tau^l_{nm}(k)=\delta_{n_0}D^l_{0m}(k), \qquad \mu=\lambda=-\tfrac{3}{2}+\mathrm{i}\sigma_2, \delta_2\in R.$$

The representation (19) is unitary with respect to the scalar product

$$\langle f|f\rangle=\sum_{lmn}|f_{lnm}|^2\rho(l,n,\mu,\lambda) \tag{20}$$

where $f(k)=\Sigma_{lnm}f_{lnm}\tau^l_{nm}(k)$. The coefficients $\rho(l,n,\mu,\lambda)$ were found by Sijacki (1975).

The representations (16), (17), (19) are operator-irreducible representations, i.e. any operator which commutes with generators of SL(N, R) is proportional to the unit operator.

For the purposes of this paper we realise the principal series of the UR of SL(N, R)($N>3$) and three series of UR of SL(2, R), and also three series of UR of SL(3, R) on the fields $\Psi_A(x)$:

$$T_g\Psi_A(x)=\Psi'_A(x)=\sum_{A'}\langle A|T(g)|A'\rangle\Psi_{A'}(g^{-1}x), \quad g\in \mathrm{SL}(N,R) \tag{21}$$

Any capital Latin index can always be replaced by: (i) a set of three lower-case indices (e.g. $(A)=\binom{\omega}{ij}$, $(A')=\binom{\omega'}{i'j'}$) for the UR (16) of SL($N$, R)($N>3$) in the $|\binom{\omega}{ij}\rangle$ basis; (ii) a set of three lower-case Latin indices (e.g. $(A)=(lnm)$, $(A'=l'n'm')$) for three series of the UR (19) of SL(3, R) in the $|lnm\rangle$ basis; (iii) a lower-case Latin index (e.g. $(A)=(m)$, $(A')=(m')$) for three series of the UR (17) of SL(2, R) in the $|m\rangle$ basis.

The representation (21) is unitary with respect to the scalar product

$$\begin{aligned}(\Psi(x),\Phi(x))&=\sum_{A,B}N_{AB}\Psi^*_A(x)\Phi_A(x)\\ (T_g\Psi(x),T_g\Phi(x))&=(\Psi(g^{-1}x),\Phi(g^{-1}x))\end{aligned} \tag{22}$$

where the coefficients N_{AB} are defined by equations (13), (18), and (20).

Let us consider now the matrix $g(\epsilon)=\{g_{ij}(\epsilon)\}\in \mathrm{SL}(N,R)$ $(i,j=1,\ldots,N)$

$$g_{ik}(\epsilon)=\delta_{ik}+\epsilon\Big(\delta_{\mu i}\delta_{\nu k}-\frac{1}{N}\delta_{\mu\nu}\delta_{ik}\Big), \qquad |\epsilon|\ll 1; \tag{23}$$

then

$$T_g\Psi_A(x)=(I+i\epsilon SF_{\mu\nu})\Psi_A(x)$$
$$=i\epsilon\Big[\sum_B \overline{(\overline{SF}_{\mu\nu})_{AB}\Psi_B(x)}+\overline{\Big(x_\nu\frac{\partial}{\partial x_\mu}-\frac{1}{N}\delta_{\mu\nu}x_\gamma\frac{\partial}{\partial x_\gamma}\Big)}\Big]\Psi_A(x)+\overline{\Psi_A(x)} \tag{24}$$

where $\overline{SF}_{\mu\nu}$ denotes the matrix representation of generator $SF_{\mu\nu}=F_{\mu\nu}-(1/N)\delta_{\mu\nu}F_{\gamma\gamma}$ in the discrete $|A\rangle$ basis.

Let us realise generators of the conformal group on the fields $\{\Psi_A(x)\}$. Such realisation is essentially defined by the representation of the translation algebra. We consider a simple well known representation of P_μ:

$$(P_\mu\Psi)_A(x)=i\frac{\partial}{\partial x_\mu}\Psi_A(x). \tag{25}$$

The dilation generator $F_{\gamma\gamma}$ commutes with the generators $R_{\mu\nu}$, $M_{\mu\nu}$ and therefore

$$(F_{\gamma\gamma}\Psi)_A(x)=d\Psi_A(x)+ix_\gamma\frac{\partial}{\partial x_\gamma}\Psi_A(x). \tag{26}$$

The scalar product (22) is invariant with respect to dilation if $d\in R$.

It is convenient to extend SL(N, R) to the algebra of the linear group GL(N, R) with generators $F_{\mu\nu}$ and

$$[F_{\mu\nu},F_{\alpha\beta}]=i(\delta_{\mu\beta}F_{\alpha\nu}-\delta_{\alpha\nu}F_{\mu\beta}). \tag{27}$$

Using the commutation relations

$$[P_\alpha,K_\beta]=-2i(\delta_{\alpha\beta}F_{\gamma\gamma}-M_{\alpha\beta}) \tag{28}$$

we can find the representations of the generators K_β on the fields $\{\Psi_A(x)\}$. Indeed using (25), (28) we obtain the following equations:

$$i\partial_\mu(K_\nu\Psi)_A(x)-(K_\nu P_\mu\Psi)_A(x)=-2i((\delta_{\mu\nu}F_{\gamma\gamma}+M_{\mu\nu})\Psi)_A(x). \tag{29}$$

Therefore

$$(K_\nu\Psi)_A(x)=-2x_\gamma(\delta_{\gamma\nu}\bar F_{\beta\beta}+\bar M_{\gamma\nu})_{AB}\Psi_B(x)+(K_\nu(0))_{AB}\Psi_B(x)$$
$$+i\Big(x^2\frac{\partial}{\partial x_\nu}-2x_\nu x_\gamma\frac{\partial}{\partial x_\gamma}\Big)\Psi_A(x) \tag{30}$$

where $(K_\nu(0))_{AB}$ is still an undetermined constant. Finally, the commutation relations

$$[F_{\gamma\gamma},K_\nu]=iK_\nu \tag{31}$$

require that the constant $(K_\nu(0))_{AB}=0$.

Similarly we can find a matrix $\bar F_{\mu\nu_1,\dots\nu_n}$ which defines a representation of the generators $F_{\mu\nu_1\nu_2\dots\nu_n}$ $(n\geqslant 2)$:

$$(F_{\mu\nu_1\dots\nu_n}\Psi)_A(x)=(F_{\mu\nu_1\dots\nu_n})_{AB}\Psi_B(x)+ix_{\nu_1}\dots x_{\nu_n}\frac{\partial}{\partial x_\mu}\Psi_A(x). \tag{32}$$

The commutation relations for $F_{\mu\nu_1\dots\nu_n}$ and P_μ can be written as

$$[P_\alpha,F_{\mu\nu_1\nu_2\dots\nu_n}]=i(\delta_{\nu_1\alpha}F_{\mu\nu_2\dots\nu_n}+\dots+\delta_{\nu_n\alpha}F_{\mu\nu_1\nu_2\dots\nu_{n-1}}). \tag{33}$$

Using (32), (33) and commutation relations

$$[F_{\gamma\gamma}, F_{\mu\nu_1\ldots\nu_n}] = \mathrm{i}(n-1)F_{\mu\nu_1\ldots\nu_n} \tag{34}$$

we obtain relations for $(\bar{F}_{\mu\nu_1\ldots\nu_n})_{AB}\Psi_B(x)$:

$$(\bar{F}_{\mu\nu_1\nu_2\ldots\nu_{n+1}})_{AB}\Psi_B(x) = (x_{\nu_1}\bar{F}_{\mu\nu_2\ldots\nu_{n+1}} + \ldots + x_{\nu_{n+1}}\bar{F}_{\mu\nu_1\ldots\nu_n})_{AB}\Psi_B(x). \tag{35}$$

Therefore generators $F_{\mu\nu_1\ldots\nu_n}$ have the compact form

$$(F_{\mu\nu_1\ldots\nu_n}\Psi)_A(x) = \frac{\partial}{\partial x_\gamma}(x_{\nu_1}\ldots x_{\nu_n})(\bar{F}_{\mu\gamma})_{AB}\Psi_B(x) + \mathrm{i}x_{\nu_1}\ldots x_{\nu_n}\frac{\partial}{\partial x_\mu}\Psi_A(x). \tag{36}$$

Using (8), (25) and (36) we obtain the following results.

Theorem. The operator-irreducible unitary representation of SL(N, R) on $\{\Psi_A(x)\}$ can be extended to a UR (with respect to the scalar product (22)) of the Lie algebra An diff R^N of the infinitesimal transformations of the An Diff R^N group. The generators $F_{\mu\gamma}$ of the algebra gL(N, R) act on the $\{\Psi_A(x)\}$ by equations

$$(F_{\mu\gamma}\Psi)_A(x) = (\bar{F}_{\mu\gamma})_{AB}\Psi_B(x) + \mathrm{i}x_\gamma\frac{\partial}{\partial x_\mu}\Psi_A(x) \tag{37}$$

where the $\bar{F}_{\mu\gamma}$ are certain matrices with constant coefficients which are determined by the original representation of SL(N, R) and by the number d appearing in (26). An arbitrary element T_f^{an} of An diff R^N acts on $\{\Psi_A(x)\}$ by the equations

$$(T_f^{\mathrm{an}}\Psi)_A(x) = (\bar{T}_f^{\mathrm{an}})_{AB}\Psi_B(x) - f_\mu^{\mathrm{an}}\frac{\partial}{\partial x_\mu}\Psi_A(x) \tag{38}$$

where the $\bar{T}_f^{\mathrm{an}}$ is the coordinate-dependent matrix

$$(\bar{T}_f^{\mathrm{an}})_{AB} = \mathrm{i}\frac{\partial f_\mu^{an}}{\partial x_\gamma}(\bar{F}_{\mu\gamma})_{AB}. \tag{39}$$

Let us define the operator $T_f(f \in U)$ on $\{\Psi_A(x)\}$:

$$T_f\Psi_A(x) = \mathrm{i}\frac{\partial f_\mu}{\partial x_\nu}(\bar{F}_{\mu\nu})_{AB}\Psi_B(x) - f_\mu\frac{\partial}{\partial x_\mu}\Psi_A(x). \tag{40}$$

That T_f is a unitary representation of diff R^N is straightforward

$$([T_f, T_h]\Psi)_A(x) = (T_\gamma\Psi)_A(x) \tag{41}$$

where

$$\gamma_\mu = h_\nu\frac{\partial}{\partial x_\nu}f_\mu - f_\nu\frac{\partial}{\partial x_\nu}h_\mu.$$

The matrix $\bar{T}_f$ $((\bar{T}_f)_{AB} = \mathrm{i}(\partial f_\mu/\partial x_\nu)(\bar{F}_{\mu\nu})_{AB}$ can be written in another form. At every point R^N having coordinates x_μ we associate to the transformation (3) the element $\Lambda(f, x)$ of the group SL(N, R) defined by the matrix $\{\Lambda_{\mu\nu}(f, x)\}$ ($\mu, \nu = 1, 2, \ldots, N$)

$$\Lambda_{\mu\nu}(f, x) = \delta_{\mu\nu} + \epsilon\left(\frac{\partial f_\mu}{\partial x_\nu} - \frac{1}{N}\delta_{\mu\nu}\frac{\partial f_\gamma}{\partial x_\gamma}\right), \qquad |\epsilon| \ll 1. \tag{42}$$

The Iwasawa decomposition takes place for $\Lambda(f, x)$ and $k \in \mathrm{SO}(N)$

$$\Lambda^{-1}(f, x)\,.\,k = k'(k, \Lambda^{-1})\,\mathrm{e}^{h(k,\Lambda^{-1})}t(k, \Lambda) \tag{43}$$

where $k'(k, \Lambda^{-1}) \in SO(N), e^{h(k,\Lambda^{-1})} \in A$, $t(k, \Lambda) \in T$. Then using (16), (17), (19), (23)–(24), (42)–(43), we obtain the matrices $\bar{T}_f$ of UR of diff R^N in the following form. For diff R^N $(N > 3)$

$$(\bar{T}_f)_{\omega' i' j', \omega i j} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\int \mathrm{d}k\, t^{*\omega'}_{i'j'}(k) t^{\omega}_{ij}(k'(k, \Lambda^{-1})) \left(\frac{\mathrm{d}k'(k, \Lambda^{-1})}{\mathrm{d}k} \right)^{1/2} \right.$$

$$\left. \times (d_\omega d_{\omega'})^{1/2} \exp[\alpha(h(k, \Lambda^{-1}))] - I_{\omega' i' j', \omega i j} \right) + \frac{\mathrm{i}}{N} \frac{\partial f_\mu}{\partial x_\mu} (\bar{F}_{\gamma\gamma})_{\omega' i' j', \omega i j}. \qquad (44a)$$

For diff R^2

$$(\bar{T}_f)_{nm} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\int \mathrm{d}k\, f^*_n(k) (a_{22}(k, \Lambda^{-1}))^{s-1} f_m(k'(k, \Lambda^{-1})) - I_{nm} \right) + \frac{\mathrm{i}}{2} \frac{\partial f_\mu}{\partial x_\mu} (\bar{F}_{\gamma\gamma})_{nm} \qquad (44b)$$

and for diff R^3

$$(\bar{T}_f)_{l'n'm', lnm}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\int \mathrm{d}k\, \tau^{*l'}_{n'm'}(k) (a_{11}(k, \Lambda))^{\mu} (a_{22}(k, \Lambda))^{\lambda} \right.$$

$$\left. \times [(2l+1)(2l'+1)]^{1/2} \tau^{l}_{nm}(k'(k, \Lambda)) - I_{l'n'm', lnm} \right) + \frac{\mathrm{i}}{3} \frac{\partial f_\mu}{\partial x_\mu} (\bar{F}_{\gamma\gamma})_{l'n'm', lnm} \qquad (44c)$$

$$k \,.\, \Lambda(f, x) = t(k, \Lambda) \,.\, a(k, \Lambda) \,.\, k'(k, \Lambda), \qquad k'(k, \Lambda) \in SO(3).$$

Using (44) we can exponentiate the representations (40) and obtain UR of the Diff R^N group. The full proof will be published elsewhere. The representation (40) for $d = \rho + \frac{1}{2}\mathrm{i}N$ $(\rho \in R)$ is unitary with respect to the scalar product

$$(\Psi, \Phi) = \int \sum_{A,B} \Psi^*_A(x) N_{AB} \Phi_B(x)\, \mathrm{d}^N x$$
$$((I + \epsilon T_f)\Psi, (I + \epsilon T_f)\Phi) = (\Psi, \Phi) \qquad |\epsilon| \ll 1. \qquad (45)$$

We demonstrate the important property of the representation (40) with the concrete example of multiplicity free (primitive) representations of the group SL(3, R). These representations are realised on the fields $\Psi_{ln}(x)$ $(-l \leq n \leq l,\ l = 0, 1, 2, \ldots, \infty)$ and

$$(\bar{M}_1 \pm \mathrm{i}\bar{M}_2)_{ln, l'n'} \Psi_{l'n'}(x) = [(l \mp n)(l \pm n + 1)]^{1/2} \Psi_{ln\pm 1}(x)$$
$$(\bar{M}_3)_{ln, l'n'} \Psi_{l'n'}(x) = n \Psi_{ln}(x) \qquad (46)$$

where $\bar{M}_i = \epsilon_{ikl} \bar{M}_{kl}$. Dothan *et al* (1966) and Biedenharn *et al* (1972) suggested that primitive representations of group SL(3, R) furnished an algebraic model of Regge poles. The linear combinations of the generators $R_{\mu\nu}$ form second rank irreducible tensor operator Q^2_p $(p = 2, 1, 0, -1, -2)$ with respect to the group SO(3) and

$$(\bar{Q}^2_p)_{ln, l'n'} \Psi_{l'n'}(x)$$

$$= \sum_\alpha \left(\frac{2l+1}{2(l+\alpha)+1} \right)^{1/2} \frac{2\mathrm{i}}{6^{1/2}} C^{l+\alpha l 0}_{l,2;0,0} \left[\mathrm{i}\sigma_2 - \alpha\left(l - \frac{1+\alpha}{2} \right) \right] C^{l+\alpha, n+p}_{l,2;n,p} \Psi_{l+\alpha, n+p}(x) \qquad (47)$$

where $C^{l,n_1+n_2}_{l_1,2;n_1,n_2}$ is Clebsch–Gordan coefficient, $\sigma_2 \in R$, $\alpha = 2, 0, -2$. Adding dilation operator $F_{\gamma\gamma}$ to sL(3, R):

$$(\bar{F}_{\gamma\gamma})_{ln,l'n'}\Psi_{l'n'}(x) = (\rho + \tfrac{3}{2}\mathrm{i})\Psi_{ln}(x) \tag{48}$$

we obtain the UR of the group GL(3, R), if $\rho \in R$ and

$$(\Phi, \Psi) = \int \mathrm{d}^3x \sum_{l=0}^{\infty} \sum_{n=-l}^{l} \Phi^*_{ln}(x)\Psi_{ln}(x). \tag{49}$$

According to § 2, the fields $\{\Psi_{ln}(x)\}$ form the basis of the UR of diff R^3 and

$$(T_f\Psi)_{ln}(x) = \mathrm{i} \sum_{l'n'} \frac{\partial f_\mu}{\partial x_\rho}(\bar{F}_{\mu\rho})_{ln,l'n'}\Psi_{l'n'}(x) - f_\mu \frac{\partial}{\partial x_\mu}\Psi_{ln}(x). \tag{50}$$

The operators T_f (50) have the commutation relations of diff R^3 not for real but complex values parameters σ_2, ρ. Let us suppose that parameters σ_2, ρ are complex numbers in the representation (50). Then this representation defines infinite-dimensional non-unitary representation of diff R^3. It is interesting to point out, that such non-unitary representation has the invariant subspaces for definite values of complex parameters σ_2, ρ. For example, if $\sigma_2 = \mathrm{i}$, $\rho = -\frac{5}{2}\mathrm{i}$ the fields $\Psi_{1n}(x)$ transform according to (50) as the covariant vector.

3. The UR of diff R^N and the non-linear realisation of diff R^N

There is a close relationship between the finite-dimensional representations of GL(N, R) and the non-linear realisation of GL(N, R) symmetry (Borisov and Ogievetsky 1974, Cho and Freund 1975). In this section we find the relationship between the non-linear realisation of diff R^N and UR of diff R^N.

Let us examine the non-linear realisation of diff R^N so that only the sO(N) algebra will be represented by linear homogeneous transformations of fields. Generalising the method proposed by Borisov and Ogievetsky (1974), Cho and Freund (1975) we introduce the symmetric tensor field $h_{\mu\nu}(x)$ ($\mu, \nu = 1, 2, \ldots, N$). The field $h_{\mu\nu}(x)$ is the gravitational field in the four-dimensional space–time ($x_1, x_2, x_3, x_4 = \mathrm{i}ct$). We define the infinitesimal transformation law of $h_{\mu\nu}$ under Diff R^N group as follows:

$$\Big(I + \mathrm{i}\epsilon\frac{\partial}{\partial x_\gamma} f_\mu \bar{F}_{\mu\gamma}\Big)_{AB}\Big(\exp\frac{\mathrm{i}}{2}h_{\mu\nu}(x)\bar{K}_{\mu\nu}\Big)_{BC}$$

$$= \Big(\exp\frac{\mathrm{i}}{2}h_{\mu\nu}(x)\bar{K}_{\mu\nu}\Big)_{AE}\Big(\exp\frac{\mathrm{i}}{2}u_{\mu\nu}(h, f)\bar{M}_{\mu\nu}\Big)_{EC} \tag{51}$$

where $x'_\mu = x_\mu + \epsilon f_\mu(x)$, $\bar{K}_{\mu\nu} = \bar{F}_{\mu\nu} + \bar{F}_{\nu\mu}$, $h'_{\mu\nu}(x')$ is the transformed field $h_{\mu\nu}$, I is the unit matrix in the $|A\rangle$ basis and $(\exp\frac{1}{2}\mathrm{i}\, h_{\mu\nu}\bar{K}_{\mu\nu})_{AB} = I_{AB} + \frac{1}{2}\mathrm{i}\, h_{\mu\nu}(\bar{K}_{\mu\nu})_{AB} + \ldots$. The infinitesimal transformation law of field $\Psi_A(x)$ under Diff R^N is defined as follows:

$$\Psi'_A(x') = \Big(\exp\frac{\mathrm{i}}{2}u_{\mu\nu}(h, f)\bar{M}_{\mu\nu}\Big)_{AB}\Psi_B(x) \tag{52}$$

In the lowest order in ϵ the fields $h_{\mu\nu}$ and $u_{\mu\nu}$ have the forms

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) + \epsilon \sum_{mn} (f_{mn}h^m(x) f h^n(x))_{\mu\nu} + \frac{\epsilon}{2}\Big[h_{\mu\gamma}(x)\Big(\frac{\partial f_\nu}{\partial x_\gamma} - \frac{\partial f_\gamma}{\partial x_\nu}\Big) + (\mu \leftrightarrow \nu)\Big] \tag{53}$$

$$u_{\mu\nu}(f,h)=\frac{\epsilon}{2}\left(\frac{\partial f_\mu}{\partial x_\nu}-\frac{\partial f_\nu}{\partial x_\mu}\right)+\sum_{mn} c_{mn}(h^m(x)fh^n(x))_{\mu\nu} \tag{54}$$

where

$$(h^m(x)fh^n(x))_{\mu\nu}=\tfrac{1}{2}h_{\mu\nu_1}\ldots h_{\nu_{m-1}\nu_m}\left(\frac{\partial f_{\nu_m}}{\partial x_{\rho_1}}+\frac{\partial f_{\rho_1}}{\partial x_{\nu_m}}\right)h_{\rho_1\rho_2}\ldots h_{\rho_n\nu} \tag{55}$$

and the coefficients b_{mn} and c_{mn} are given by the generating functions

$$\begin{aligned}G_1(x,y)&=\sum_{mn} b_{mn}x^m y^n=(x-y)\coth(x-y)\\ G_2(x,y)&=\sum_{mn} c_{mn}x^m y^n=\tanh\frac{x-y}{2}.\end{aligned} \tag{56}$$

Letting

$$\begin{aligned}\delta_f h_{\mu\nu}&=\frac{1}{\epsilon}(h'_{\mu\nu}(x')-h_{\mu\nu}(x))\\ \delta_f\bar{\Psi}_A(x)&=\frac{1}{2\epsilon \mathrm{i}}u_{\mu\nu}(h,f)(\bar{M}_{\mu\nu})_{AB}\bar{\Psi}_B(x)\end{aligned} \tag{57}$$

we find that

$$\begin{aligned}(\delta_f\delta_h-\delta_h\delta_f)h_{\mu\nu}&=\delta_\gamma h_{\mu\nu}\\ (\delta_f\delta_h-\delta_h\delta_f)\bar{\Psi}_A(x)&=\delta_\gamma\Psi_A(x)\end{aligned} \tag{58}$$

where $f, h\in u$, $\gamma_\mu=h_\nu(\partial f_\mu/\partial x_\nu)-f_\nu(\partial h_\mu/\partial x_\nu)$. Hence the transformations (57) give the non-linear realisation of diff R^N.

It is possible to construct the functions of $h_{\mu\nu}$ which transform linearly. These quantities are represented by the squares of $(\exp -h)_{\mu\nu}=\delta_{\mu\nu}-h_{\mu\nu}+\ldots$ and $(\exp h)_{\mu\nu}$

$$\begin{aligned}g^{\mu\nu}&=(\exp 2h)_{\mu\nu}; &\qquad \delta_f g^{\mu\nu}&=g^{\mu\gamma}\frac{\partial f_\nu}{\partial x_\gamma}+g^{\nu\gamma}\frac{\partial f_\mu}{\partial x_\gamma}\\ g_{\mu\nu}&=(\exp -2h)_{\mu\nu}; &\qquad \delta_f g_{\mu\nu}&=-g_{\mu\gamma}\frac{\partial f_\gamma}{\partial x_\nu}-g_{\nu\gamma}\frac{\partial f_\gamma}{\partial x_\mu}\end{aligned} \tag{59}$$

and correspond to the contravariant and covariant metric tensors in the general relativity. By changing the fields $\bar{\Psi}_A(x)$ it is possible to introduce the linearly transformed fields $\Psi_A(x)$. From (51) and (52) we obtain that under the action of Diff R^N group the fields

$$\Psi_A(x)=\left(\exp\frac{\mathrm{i}}{2}h_{\mu\nu}\bar{K}_{\mu\nu}\right)_{AB}\Psi_B(x) \tag{60}$$

are transformed as follows

$$\begin{aligned}\Psi'_A(x')&=\left(I+\mathrm{i}\epsilon\frac{\partial f_\mu}{\partial x_\nu}\bar{F}_{\mu\nu}\right)_{AB}\Psi_B(x)\\ x'_\mu&=x_\mu+\epsilon f_\mu(x),\qquad |\epsilon|\ll 1.\end{aligned} \tag{61}$$

The fields $\bar{\Psi}_A(x)$ are useful in the construction of interaction Lagrangians of $h_{\mu\nu}$ with the fields $\Psi_A(x)$. Such Lagrangians will be discussed elsewhere.

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